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Ideal convergence characterization of the completion of linear n-normed spaces

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ABSTRACT

In this paper we give the completion of a linear n-normed space in terms of ideal convergence by introducing the concept of a uniformly continuous n-norm.

1. Introduction

Kostyrko et al. (cf. [1]; a similar concept was invented in [2]) introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence.

The notion of statistical convergence of sequences of real numbers was introduced by Fast in [3] and Steinhaus in [4] (see also [5]) and is based on the notion of asymptotic density of a set $A \subset \mathbb{N}$ [6–8]. However, the first idea of statistical convergence appeared (under the name almost convergence) in the first edition (Warsaw, 1935) of the celebrated monograph [9] of Zygmund. It should also be mentioned that the notion of statistical convergence has been considered in other contexts, by several people like Bernstein, Frolik, etc.

Statistical convergence had been discussed and developed by many authors including Connor [10], Erdös and Tenenbaum [11], Freedman and Sember [12], Fridy [13], Miller [14], Pehlivan and Karaev [15], Pehlivan et al. [16], Di Maio and Kočiniac [17], Balcerzak et al. [18].

The concepts of 2-metric spaces and 2-normed spaces were initially introduced by Gähler [19–21] in 1960's. This notion which is nothing but a two dimensional analogue of a normed space drew the attention of a wider audience after the publication of a paper by George [22]. Siddiqi delivered a series of lectures on this theme in various conferences. His joint paper with Gähler et al. [23] also provided valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by Siddiqi [24]. Since then, many researchers have studied these subjects and obtained various results [25–33].

In this paper we study the completion of a linear n-normed space via ideal convergence by introducing the concept of uniformly continuous n-norm and ideal equivalent sequences.

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First we introduce some notation. Throughout this paper $\mathbb{N}$ will denote the set of positive integers. If K is a subset of $\mathbb{N}_0$, the set of natural numbers, then K_n denotes the set $\{k \leq n : k \in K\}$. The natural density of K, denoted by $\delta(K)$ is given by

$$\delta(K) = \lim \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists, where $|K_n|$ denotes the cardinality of the set K_n [12].

Let $(X, \|.\|)$ be a normed space. Recall that a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of X is said to be statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x\| \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$.

A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (i) $\emptyset \in \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; (iii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ [34,35].

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal (i.e., $\mathcal{I} \neq \emptyset$ and $Y \notin \mathcal{I}$) in $\mathbb{N}$. The sequence $(x_k)_{k\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [1,36].

The sequence $\{x_k\}$ is said to be an *$\mathcal{I}$-Cauchy sequence* if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $\left\{k \in \mathbb{N} : \left\|x_k - x_{N(\varepsilon)}\right\| \geq \varepsilon\right\}$ belongs to $\mathcal{I}$ [37,38].

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies the following four conditions:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x, y\| = \|y, x\|$;
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|, \alpha \in \mathbb{R}$;
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|.,.\|)$ is then called a 2-normed space [19,20].

The sequence $\{x_k\}$ in a 2-normed space $(X, \|.,.\|)$ is said to be convergent to L in X if $\lim_{k\to\infty} \|x_k - L, z\| = 0$ for every $z \in X$. In such a case, we write $\lim_{k\to\infty} \|x_k, z\| := \|L, z\|$ [39].

The sequence $\{x_k\}$ in a 2-normed space $(X, \|.,.\|)$ is said to be Cauchy sequence in X if $\lim_{m,k\to\infty} \|x_m - x_k, z\| = 0$ for every $z \in X$ [39].

In 2007, Şahiner et al. gave the following definitions in [33].

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence (x_n) in a 2-normed space X is said to be $\mathcal{I}$-convergent to x, if for each $\varepsilon > 0$ and z in X the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x, z\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

Further we will give some examples of ideals and corresponding $\mathcal{I}$-convergences.

(I) Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ convergence coincides with usual convergence [19].

(II) Put $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_\delta$ convergence coincides with the statistical convergence [29].

(III) A wide class of $\mathcal{I}$-convergences can be obtained as follows. Let $T = \left\{t_{n,k}\right\}_{n,k\in\mathbb{N}}$ be a regular non negative matrix. For $A \subset \mathbb{N}$ we put $\delta_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{n,k}\chi_A(k)$ for $n \in \mathbb{N}$. If $\lim_{n\to\infty} \delta_T^{(n)}(A) = \delta_T(A)$ exists, then $\delta_T(A)$ is called a T-density of A (see [14]). Put $\mathcal{I}_{\delta_T} = \{A \subset \mathbb{N} : \delta_T(A) = 0\}$. Then $\mathcal{I}_{\delta_T}$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_{\delta_T}$ contains $\mathcal{I}_\delta$-convergence as special cases (see [1]).

In [1] another special case of $\mathcal{I}_{\delta_T}$-convergence is given.

2. Preliminaries

In 2001, Gunawan and Mashadi in [27] gave the following definitions.

Definition 1. Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$ (here we allow d to be infinite). A real-valued function $\|., \ldots, .\|$ on X^n satisfying the following four properties

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;
(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;
(iii) $\|x_1, x_2, \ldots, x_{n-1}, \alpha x_n\| = |\alpha| \|x_1, x_2, \ldots, x_{n-1}, x_n\|$, for any $\alpha \in \mathbb{R}$;
(iv) $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$,

is called an n-norm on X and the pair $(X, \|., \ldots, .\|)$ is called an n-normed space.

It is well-known fact from the following corollary that n-normed spaces are actually normed spaces (see also [40]).

Corollary 1 (*[27]*)**.** *Every n-normed space is an $(n - r)$-normed space for all $r = 1, \ldots, n - 1$. In particular, every n-normed space is a normed space.*

Example 1. A standard example of an n-normed space is $X = \mathbb{R}^n$ equipped with the n-norm is

$\|x_1, x_2, \ldots, x_{n-1}, x_n\| :=$ the volume of the n-dimensional parallelepiped spanned by $x_1, x_2, \ldots, x_{n-1}, x_n$ in X.

More examples of n-normed spaces can be found [27,28].

Observe that in any n-normed space $(X, \|., \dots, .\|)$ we have

$$\|x_1, x_2, \dots, x_{n-1}, x_n\| \geq 0$$

and

$$\|x_1, x_2, \dots, x_{n-1}, x_n\| = \|x_1, x_2, \dots, x_{n-1}, x_n + \alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}\|$$

for all $x_1, x_2, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}$.

Definition 2. A sequence $\{x_k\}$ in an n-normed space $(X, \|., \dots, .\|)$ is said to be convergent to an $l \in X$ if

$$\lim_{k\to\infty} \|x_k - l, z_1, z_2, \dots, z_{n-1}\| = 0$$

for every $z_1, z_2, \dots, z_{n-1} \in X$.

Definition 3. A sequence $\{x_k\}$ in an n-normed space $(X, \|., \dots, .\|)$ is called a Cauchy sequence if

$$\lim_{k,l\to\infty} \|x_k - x_l, z_1, z_2, \dots, z_{n-1}\| = 0$$

for every $z_1, z_2, \dots, z_{n-1} \in X$.

Now, we introduce the notion of ideal convergence in n-normed spaces and give the main results of the paper.

3. Main results

Suppose hereafter that $\mathcal{I}$ is a nontrivial ideal in $2^{\mathbb{N}}$, unless otherwise stated.

Definition 4. Let $\{x_k\}$ be a sequence in an n-normed space $(X, \|., \dots, .\|)$. Then the sequence $\{x_k\}$ is said to be ideal convergent to L if for every $\varepsilon > 0$ and any nonzero $z_1, z_2, \dots, z_{n-1}$ in X the set

$$\{k \in \mathbb{N} : \|x_k - L, z_1, z_2, \dots, z_{n-1}\| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. In this case we write

$$\mathcal{I}\text{-}\lim_{k\to\infty} X_{(x_k)} = L.$$

Definition 5. A sequence $\{x_k\}$ in a linear n-normed space $(X, \|., \dots, .\|)$ is said to be an ideal Cauchy sequence in X if for every $\varepsilon > 0$ and nonzero $z_1, z_2, \dots, z_{n-1} \in X$ there exists a number $N = N(\varepsilon, z_1, z_2, \dots, z_{n-1})$ such that

$$\{k \in \mathbb{N} : \|x_k - x_m, z_1, z_2, \dots, z_{n-1}\| \geq \varepsilon\} \in \mathcal{I},$$

for all $m \geq N$.

Theorem 1. *Let $\{x_k\}$ and $\{y_k\}$ be any two ideal Cauchy sequences in an n-normed space. Then so are $\{x_k + y_k\}$ and $\{\lambda x_k\}$, where λ is a scalar.*

Proof. If $\{x_k\}$ and $\{y_k\}$ are ideal Cauchy sequences in $(X, \|., \dots, .\|)$, then for every $\varepsilon > 0$ and $z_1, z_2, \dots, z_{n-1} \in X$ there exists a number $N = N(\varepsilon, z_1, z_2, \dots, z_{n-1})$ such that

$$K_1 = \left\{k \in \mathbb{N} : \|x_k - x_m, z_1, z_2, \dots, z_{n-1}\| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}$$

and

$$K_2 = \left\{k \in \mathbb{N} : \|y_k - y_m, z_1, z_2, \dots, z_{n-1}\| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}$$

for all $m \geq N$. Let

$$K = K(\varepsilon) = \{k \in \mathbb{N} : \|(x_k + y_k) - (x_m + y_m), z_1, z_2, \dots, z_{n-1}\| \geq \varepsilon\}.$$

Since it is absolutely straightforward to prove that $K \subset K_1 \cup K_2$, we omit it.

Now, it is also routine to show that if $\{x_k\}$ is an $\mathcal{I}$-Cauchy sequence then so is $\{\lambda x_k\}$. □

Definition 6. Two ideal Cauchy sequences $\{x_k\}$ and $\{y_k\}$ in a linear n-normed space $(X, \|., \dots, .\|)$ are said to be ideal equivalent if for every neighborhood U of 0 there is an integer $N(U)$ such that

$$\{k \in \mathbb{N} : k \geq N(U) \text{ and } x_k - y_k \notin U\} \in \mathcal{I}.$$

This is equivalent to

$$\mathcal{I}\text{-}\lim_{k\to\infty} \|x_k - y_k, z_1, z_2, \ldots, z_{n-1}\| = 0$$

for every $z_1, z_2, \ldots, z_{n-1} \in X$. If $\{x_k\}$ and $\{y_k\}$ are ideal equivalent, then write $\{x_k\} \overset{\mathcal{I}}{\approx} \{y_k\}$.

Remark 1. Let $(X, \|., \ldots, .\|)$ be a linear n-normed space. If $\{a_k\} \overset{\mathcal{I}}{\approx} \{x_k\}$ and $\{b_k\} \overset{\mathcal{I}}{\approx} \{y_k\}$, then $\{a_k + b_k\} \overset{\mathcal{I}}{\approx} \{x_k + y_k\}$ and $\{\alpha a_k\} \overset{\mathcal{I}}{\approx} \{\alpha x_k\}$ with $\alpha \in \mathbb{R}$.

Proof. Since $\{a_k\} \overset{\mathcal{I}}{\approx} \{x_k\}$ and $\{b_k\} \overset{\mathcal{I}}{\approx} \{y_k\}$, for any $\varepsilon > 0$ and $z_1, z_2, \ldots, z_{n-1} \in X$ we have $K_1, K_2 \in \mathcal{I}$ where

$$K_1 = \left\{k \in \mathbb{N} : \|a_k - x_k, z_1, z_2, \ldots, z_{n-1}\| \geq \frac{\varepsilon}{2}\right\}$$

and

$$K_2 = \left\{k \in \mathbb{N} : \|b_k - y_k, z_1, z_2, \ldots, z_{n-1}\| \geq \frac{\varepsilon}{2}\right\}.$$

Let

$$K = K(\varepsilon) = \{k \in \mathbb{N} : \|(a_k + b_k) - (x_k + y_k), z_1, z_2, \ldots, z_{n-1}\| \geq \varepsilon\}.$$

Since the proofs of $K \subset K_1 \cup K_2$ and $\{\alpha a_k\} \overset{\mathcal{I}}{\approx} \{\alpha x_k\}$ are routine, we omit the detail. □

We denote the set of all equivalence classes of ideal Cauchy sequences in X by $\widehat{X}$. Let $\widehat{x}, \widehat{y}$ etc. denote the elements of $\widehat{X}$. Define an addition and scalar multiplication on $\widehat{X}$ as follows:

(i) $\widehat{x} + \widehat{y}$ =the set of all sequences ideal equivalent to $\{x_k + y_k\}$, where $\{x_k\} \in \widehat{x}$ and $\{y_k\} \in \widehat{y}$.
(ii) $\alpha\widehat{x}$ =the set of all sequences ideal equivalent to $\{\alpha x_k\}$, where $\{x_k\} \in \widehat{x}$.

By Remark 1, these two operations are well-defined since they are independent of the choice of elements from $\widehat{x}$ and $\widehat{y}$. With these two operations, $\widehat{X}$ is a linear space.

Remark 2. If $\{x_k\}$ is an ideal Cauchy sequence in a linear n-normed space $(X, \|., \ldots, .\|)$ and $z_1, z_2, \ldots, z_{n-1} \in X$, then

$$\mathcal{I}\text{-}\lim_{k\to\infty} \{\|x_k, z_1, z_2, \ldots, z_{n-1}\|\}$$

exists.

In the proof of Remark 2, similar arguments as in [37] such as the completeness of $\mathbb{R}$ and some known facts are used.

Theorem 2. *If two ideal Cauchy sequences $\{x_{1(k)}\}$ and $\{x_{2(k)}\}$ in a linear n-normed space $(X, \|., \ldots, .\|)$ are ideal equivalent, then*

$$\mathcal{I}\text{-}\lim_{k\to\infty} \left\|x_{1(k)}, z_1, z_2, \ldots, z_{n-1}\right\| = \mathcal{I}\text{-}\lim_{k\to\infty} \left\|x_{2(k)}, z_1, z_2, \ldots, z_{n-1}\right\|$$

for every $z_1, z_2, \ldots, z_{n-1} \in X$.

Proof. Since

$$\begin{aligned} \left\|x_{1(k)}, z_1, z_2, \ldots, z_{n-1}\right\| &= \left\|x_{1(k)} - x_{2(k)} + x_{2(k)}, z_1, z_2, \ldots, z_{n-1}\right\| \\ &\leq \left\|x_{1(k)} - x_{2(k)}, z_1, z_2, \ldots, z_{n-1}\right\| + \left\|x_{2(k)}, z_1, z_2, \ldots, z_{n-1}\right\| \end{aligned}$$

we have

$$\left\|x_{1(k)}, z_1, z_2, \ldots, z_{n-1}\right\| - \left\|x_{2(k)}, z_1, z_2, \ldots, z_{n-1}\right\| \leq \left\|x_{1(k)} - x_{2(k)}, z_1, z_2, \ldots, z_{n-1}\right\|$$

and

$$\left\|x_{2(k)}, z_1, z_2, \ldots, z_{n-1}\right\| - \left\|x_{1(k)}, z_1, z_2, \ldots, z_{n-1}\right\| \leq \left\|x_{2(k)} - x_{1(k)}, z_1, z_2, \ldots, z_{n-1}\right\|.$$

Thus we obtain the following

$$\left\{k \in \mathbb{N} : \left|\left\|x_{2(k),} z_1, \ldots, z_{n-1}\right\| - \left\|x_{1(k)}, z_1, z_2, \ldots, z_{n-1}\right\|\right| \geq \varepsilon\right\} \subseteq \left\{k \in \mathbb{N} : \left\|x_{2(k)} - x_{1(k)}, z_1, \ldots, z_{n-1}\right\| \geq \varepsilon\right\}.$$

Since $\{x_{1(k)}\}$ and $\{x_{2(k)}\}$ are ideal equivalent

$$\left\{k \in \mathbb{N} : \left\|x_{2(k)} - x_{1(k)}, z_1, \ldots, z_{n-1}\right\| \geq \varepsilon\right\} \in \mathcal{I}$$

and consequently the following completes the proof.

$$\left\{k \in \mathbb{N} : \left|\left\|x_{2(k),} z_1, \ldots, z_{n-1}\right\| - \left\|x_{1(k)}, z_1, z_2, \ldots, z_{n-1}\right\|\right| \geq \varepsilon\right\} \in \mathcal{I}. \quad \square$$

One of the major problems in working towards the completion of a linear n-normed space has been the question of the existence of $\mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)},x_{2(k)},\ldots,x_{n(k)}\right\|$ for any $n\mathcal{I}$-Cauchy sequences in X. The problem is solved for spaces which have uniformly continuous n-norms defined on them. As will be shown subsequently, such spaces can be completed at least to a pseudo n-normed space.

Definition 7. An n-norm $\|.,\ldots,.\|$ defined on a linear space X is said to be ideal uniformly continuous in each of these n variables if for every $\varepsilon > 0$ there exists a neighborhood U_ε of 0 such that for any choice $x_{1i}, x_{2i}, \ldots, x_{ni}, x_{1j}, x_{2j}, \ldots, x_{nj}$ with $x_{1i} - x_{1j},\ x_{2i} - x_{2j}, \ldots, x_{ni} - x_{nj} \in U_\varepsilon\ \ (1 \le i,\ j \le n)$ we have

$$\left\{k \in \mathbb{N} : \left|\|x_{1i}, x_{2i}, \ldots, x_{ni}\| - \left\|x_{1j}, x_{2j}, \ldots, x_{nj}\right\|\right| \ge \varepsilon\right\} \in \mathcal{I}.$$

Definition 8. A pseudo n-norm is defined to be a real function having all the properties of an n-norm except the condition that $\|x_1, x_2, \ldots, x_n\| = 0$ implies the linear dependence of $x_1, x_2, \ldots, x_n$.

We merely state the following lemmas and omit the proofs:

Lemma 1. *If X is a linear space having an ideal uniformly continuous n-norm, then for any n $\mathcal{I}$-Cauchy sequences $\left\{x_{1(k)}\right\}$, $\left\{x_{2(k)}\right\}, \ldots, \left\{x_{n(k)}\right\}$ in X*

$$\mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)},x_{2(k)},\ldots,x_{n(k)}\right\|$$

exists.

The proof of Lemma 1 can be proved by using Definition 7 and similar arguments as in the proof of Remark 2.

Lemma 2. *If X has an ideal uniformly continuous n-norm, then for pairs of equivalent ideal Cauchy sequences,*

$$\left\{x_{1(k)}\right\} \overset{\mathcal{I}}{\approx} \left\{a_{1(k)}\right\}, \left\{x_{2(k)}\right\} \overset{\mathcal{I}}{\approx} \left\{a_{2(k)}\right\}, \ldots, \left\{x_{n(k)}\right\} \overset{\mathcal{I}}{\approx} \left\{a_{n(k)}\right\},$$

we have

$$\mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)},x_{2(k)},\ldots,x_{n(k)}\right\| = \mathcal{I}\text{-}\lim_{k\to\infty}\left\|a_{1(k)},a_{2(k)},\ldots,a_{n(k)}\right\|.$$

Now if X is a linear space with a uniformly continuous n-norm, then for every $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n \in \widehat{X}$ we can define a real valued function on the space $\widehat{X}$ by the following:

$$\|\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\| = \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)}, x_{2(k)}, \ldots, x_{n(k)}\right\|,$$

where $\left\{x_{1(k)}\right\} \in \widehat{x}_1$, $\left\{x_{2(k)}\right\} \in \widehat{x}_2, \ldots, \left\{x_{n(k)}\right\} \in \widehat{x}_n$. The function is well defined since the limit exists by Lemma 1 and it is independent of the choice of elements in $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n$ by Lemma 2. The most that can be said thus far regarding the function defined above is that it is a pseudo n-norm for the space $\widehat{X}$. It remains an open question whether the function is a n-norm.

Theorem 3. *If X is a linear space with an ideal uniformly continuous n-norm and $\left\{x_{1(k)}\right\} \in \widehat{x}_1$, $\left\{x_{2(k)}\right\} \in \widehat{x}_2, \ldots, \left\{x_{n(k)}\right\} \in \widehat{x}_n$ are ideal Cauchy sequences then the function*

$$\|\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\| = \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)}, x_{2(k)}, \ldots, x_{n(k)}\right\|$$

is a pseudo n-norm on $\widehat{X}$.

Proof. (i) If $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n$ are linearly dependent then we may write

$$\begin{aligned}\|\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\| &= \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)}, x_{2(k)}, \ldots, x_{n(k)}\right\| \\ &= 0.\end{aligned}$$

(ii) Since

$$\|\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\| = \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)}, x_{2(k)}, \ldots, x_{n(k)}\right\|$$

and $\left\|x_{1(k)}, x_{2(k)}, \ldots, x_{n(k)}\right\|$ is invariant under permutation, so is $\|\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\|$.
(iii)

$$\begin{aligned}\|\widehat{x}_1, \widehat{x}_2, \ldots, \alpha\widehat{x}_n\| &= \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)}, x_{2(k)}, \ldots, \alpha x_{n(k)}\right\| \\ &= \mathcal{I}\text{-}\lim_{k\to\infty}|\alpha|\left\|x_{1(k)}, x_{2(k)}, \ldots, x_{n(k)}\right\| \\ &= |\alpha|\,\|\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\|.\end{aligned}$$

(iv)

$$\begin{aligned}\|\widehat{x}_1,\widehat{x}_2,\ldots\widehat{x}_{n-1},\widehat{y}+\widehat{z}\| &= \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)},x_{2(k)},\ldots,x_{(n-1)(k)},(y+z)_{(k)}\right\| \\ &\le \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)},x_{2(k)},\ldots,x_{(n-1)(k)},y_{(k)}\right\| + \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)},x_{2(k)},\ldots,x_{(n-1)(k)},z_{(k)}\right\| \\ &= \|\widehat{x}_1,\widehat{x}_2,\ldots\widehat{x}_{n-1},\widehat{y}\| + \|\widehat{x}_1,\widehat{x}_2,\ldots\widehat{x}_{n-1},\widehat{z}\|. \quad \square\end{aligned}$$

Let $\widehat{X}_0$ be a subset of $\widehat{X}$ consisting of those equivalence classes which contains an ideal Cauchy sequence $\{x_k\}$ for which $x_1 = x_2 = \cdots = x_k = \cdots$. At most one sequence of this kind can be in each ideal equivalence class. If $\widehat{x}_1,\widehat{x}_2,\ldots,\widehat{x}_n$ are in $\widehat{X}_0$ and if the corresponding Cauchy sequences are $\left\{x_{1(k)}\right\},\left\{x_{2(k)}\right\},\ldots,\left\{x_{n(k)}\right\}$ with $x_{1(k)} = x_1, x_{2(k)} = x_2,\ldots,x_{n(k)} = x_n$ for every k, then we have

$$\begin{aligned}\|\widehat{x}_1,\widehat{x}_2,\ldots,\widehat{x}_n\| &= \mathcal{I}\text{-}\lim_{k\to\infty}\left\|x_{1(k)},x_{2(k)},\ldots,x_{n(k)}\right\| \\ &= \mathcal{I}\text{-}\lim_{k\to\infty}\|x_1,x_2,\ldots,x_n\| \\ &= \|x_1,x_2,\ldots,x_n\|.\end{aligned}$$

Thus $\widehat{X}_0$ and X are isometric.

We say that $Y \subseteq \widehat{X}$ is ideal dense if for each $\widehat{x}\in\widehat{X}\setminus Y$ there is a sequence $\left\{\widehat{x}^k\right\}\subseteq Y$ such that

$$\mathcal{I}\text{-}\lim_{k\to\infty}\left\|\widehat{x}^k-\widehat{x},\widehat{z}_1,\ldots,\widehat{z}_{n-1}\right\| = 0$$

for all $\widehat{z}_1,\ldots,\widehat{z}_{n-1}\in\widehat{X}$.

Theorem 4. *If X is a linear space with an ideal uniformly continuous n-norm, then $\widehat{X}_0$ is ideal dense in $\widehat{X}$.*

Proof. Let $\widehat{x}$ be an arbitrary element in $\widehat{X}-\widehat{X}_0$ and $\{x_k\}$ be an ideal Cauchy sequence in X such that $\{x_k\}\in\widehat{x}$. For each k, $\widehat{x}^k$ is defined to be the element in $\widehat{X}_0$ which contains the repetitive sequence $x_k, x_k, \ldots$. For each $\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}$ in $\widehat{X}$ and each $\left\{z_{(1)k}\right\}\in\widehat{z}_1, \left\{z_{(2)k}\right\}\in\widehat{z}_2,\ldots,\left\{z_{(n-1)k}\right\}\in\widehat{z}_{n-1}$, since the n-norm is ideal uniformly continuous we have

$$\left\{k\in\mathbb{N}:\left\|\left(x^k\right)_m - x_m, z_{1(m)},\ldots,z_{(n-1)(m)}\right\|\ge\varepsilon\right\} = \left\{k\in\mathbb{N}:\left\|x_k-x_m,z_{1(m)},\ldots,z_{(n-1)(m)}\right\|\ge\varepsilon\right\}\in\mathcal{I}$$

where $\left(x^k\right)_m$ is the mth-component of the repetitive sequence in $\widehat{x}^k$. It is clear that for all $k\ge N(\varepsilon,z_1,z_2,\ldots,z_{n-1})$

$$\left\{k\in\mathbb{N}:\left\|\left(\widehat{x}^k\right)-\widehat{x},\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\right\|\ge\varepsilon\right\}\in\mathcal{I}$$

and hence $\mathcal{I}\text{-}\lim_{k\to\infty}\left\|\widehat{x}^k-\widehat{x},\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\right\| = 0$ for each $\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\in\widehat{X}$. Thus $\widehat{X}_0$ is ideal dense in $\widehat{X}$. $\square$

We say that a pseudo n-normed space X is ideal complete if every ideal Cauchy sequence is ideal convergent in X.

Theorem 5. *If X is a linear space having an ideal uniformly continuous n-norm, then $\widehat{X}$ is ideal complete.*

Proof. Let $\{\widehat{y}_k\}$ be an ideal Cauchy sequence in $\widehat{X}$. Let $\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}$ be any arbitrary elements in $\widehat{X}$ and for each k, from the fact that $\widehat{X}_0$ is ideal dense in $\widehat{X}$ we can choose $\widehat{w}_k$ in $\widehat{X}_0$ such that

$$\left\{k\in\mathbb{N}:\|\widehat{y}_k-\widehat{w}_k,\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\|\ge\frac{1}{k}\right\}\in\mathcal{I}.$$

Let $\varepsilon>0$. Using (iv) property of n-norm$\|.,\ldots,.\|$, we have

$$\begin{aligned}K &:= \{k\in\mathbb{N}:\|\widehat{w}_k-\widehat{w}_m,\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\|\ge\varepsilon\} \\ &\subseteq \left\{k\in\mathbb{N}:\|\widehat{w}_k-\widehat{y}_k,\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\|\ge\frac{1}{k}\right\}\cup\left\{k\in\mathbb{N}:\|\widehat{y}_k-\widehat{y}_m,\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\|\ge\frac{\varepsilon}{3}\right\} \\ &\quad\cup\left\{m\in\mathbb{N}:\|\widehat{y}_m-\widehat{w}_m,\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\|\ge\frac{1}{m}\right\},\end{aligned}$$

for $k,m\ge N=N(\varepsilon,z_1,z_2,\ldots,z_{n-1})$ where $N\ge\frac{\varepsilon}{3}$ and the sets

$$\left\{k\in\mathbb{N}:\|\widehat{w}_k-\widehat{y}_k,\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\|\ge\frac{1}{k}\right\}$$

and

$$\left\{m\in\mathbb{N}:\|\widehat{y}_m-\widehat{w}_m,\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_{n-1}\|\ge\frac{1}{m}\right\}$$

belong to $\mathcal{I}$. Since $\widehat{z}_1, \widehat{z}_2, \ldots, \widehat{z}_{n-1}$ are arbitrary and $\widehat{y}_k$ is an ideal Cauchy sequence in $\widehat{X}$, $\{\widehat{w}_k\}$ is an ideal Cauchy sequence in $\widehat{X_0}$. Let $\{w_k\}$ be the sequence in X corresponding to $\{\widehat{w}_k\}$ under the isometry between $\widehat{X}$ and $\widehat{X_0}$. Then $\{w_k\}$ is an ideal Cauchy sequence in X. Hence there exists an element $\widehat{y}$ in $\widehat{X}$ such that $\{w_k\} \in \widehat{y}$ and

$$\begin{aligned} K_1 &:= \{k \in \mathbb{N} : \|\widehat{y}_k - \widehat{y}, \widehat{z}_1, \widehat{z}_2, \ldots, \widehat{z}_{n-1}\| \geq \varepsilon\} \\ &\subseteq \left\{k \in \mathbb{N} : \|\widehat{y}_k - \widehat{w_k}, \widehat{z}_1, \widehat{z}_2, \ldots, \widehat{z}_{n-1}\| \geq \frac{1}{k}\right\} \cup \left\{k \in \mathbb{N} : \|\widehat{w}_k - \widehat{y}, \widehat{z}_1, \widehat{z}_2, \ldots, \widehat{z}_{n-1}\| \geq \frac{\varepsilon}{2}\right\} \end{aligned}$$

for $k \geq N = N(\varepsilon, z_1, z_2, \ldots, z_{n-1})$ where $N \geq \frac{\varepsilon}{2}$. As in the proof of Theorem 4,

$$\{k \in \mathbb{N} : \|\widehat{w}_k - \widehat{y}, \widehat{z}_1, \widehat{z}_2, \ldots, \widehat{z}_{n-1}\| \geq \varepsilon\} \in \mathcal{I}$$

for $k \geq N(\varepsilon, z_1, z_2, \ldots, z_{n-1})$ provided

$$\left\{k \in \mathbb{N} : \left\| w_k - w_m, z_{1(m)}, z_{2(m)}, \ldots, z_{(n-1)(m)} \right\| \geq \varepsilon\right\} \in \mathcal{I}$$

for $k, m \geq N(\varepsilon, z_1, z_2, \ldots, z_{n-1})$. Thus we showed that $K_1 \in \mathcal{I}$. Since $\{\widehat{y}_k\}$ in $\widehat{X}$ is arbitrary, the space $\widehat{X}$ is ideal complete. □

An example of an ideal uniformly continuous n-norm $\|., \ldots, .\|$ is the function defined by

$$\|x_1, x_2, \ldots, x_n\| = \frac{1}{2}\left(\sum_{1\leq i<j<n}^{d} \begin{vmatrix} x_{1i} & x_{1(i+1)} & \cdots & x_{1j} \\ x_{2i} & x_{2(i+1)} & \cdots & x_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ x_{ni} & x_{n(i+1)} & \cdots & xnj \end{vmatrix}^2\right)^{\frac{1}{2}},$$

where $x_j = \sum_{i=1}^{n} x_{ji} e_i$, $j = 1, 2, \ldots, n$ for basis elements $\{e_i\}_{i=1}^{n}$.

For example, Gähler [20] has shown that a well defined 2-norm not only on finite dimensional spaces but also on Hilbert sequence spaces is

$$\|a, b\| = \frac{1}{2}\left(\sum_{i<j}^{n} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}^2\right)^{\frac{1}{2}},$$

where $a = \sum_{i=1}^{n} a_i e_i$ and $b = \sum_{i=1}^{n} b_i e_i$ for basis elements $\{e_i\}_{i=1}^{n}$. This is an example of a uniformly continuous 2-norm $\|., .\|$.

References

[1] P. Kostyrko, T. Salat, W. Wilczynski, $\mathcal{I}$-convergence, Real Anal. Exchange 26 (2) (2000) 669–686.
[2] M. Katětov, Products of filters, Comment. Math. Univ. Carolin. 9 (1968) 173–189.
[3] H. Fast, Sur La Convergence Statistique, Colloq. Math. 2 (1951) 241–244.
[4] H. Steinhaus, Sur La Convergence Ordinarie et la Convergence Asymptotique, Colloq. Math. 2 (1951) 73–74.
[5] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.
[6] H. Halberstam, K.F. Roth, Sequences, I, Clarendon Press, Oxford, 1966.
[7] I. Niven, H.S. Zuckerman, H. Montgomery, An Introduction to the Theory of Numbers, 5th ed., Wiley, New York, 1991.
[8] H.H. Ostmann, Additive Zahlentheorie I, Springer-Verlag, Berlin, 1956.
[9] A. Zygmund, Trigonometric Series, 2nd ed., Cambridge Univ. Press, Cambridge, 1979.
[10] J. Connor, The statistical and strong p-cesaro convergence of sequences, Analysis 8 (1988) 47–63.
[11] R. Erdös, G. Tenenbaum, Sur les densities de certaines suites d'entiers, Proc. Lond. Math. Soc. 59 (3) (1989) 417–438.
[12] A.R. Freedman, J.J. Sember, Densities and summability, Pacific J. Math. 95 (1981) 293–305.
[13] J.A. Fridy, On statistical convergence, Analysis 5 (1985) 301–313.
[14] H.I. Miller, A measure theoretical subsequence characterization of statistical convergence, Trans. Amer. Math. Soc. 347 (1995) 1811–1819.
[15] S. Pehlivan, M.T. Karaev, Some results related with statistical convergence and Berezin symbols, J. Math. Anal. Appl. 299 (2) (2004) 333–340.
[16] S. Pehlivan, M. Gürdal, B. Fisher, Lacunary statistical cluster points of sequences, Math. Commun. 11 (2006) 39–46.
[17] G. Di Maio, Lj.D.R. Kočiniac, Statistical convergence in topology, Topology Appl. 156 (2008) 28–45.
[18] M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (2007) 715–729.
[19] S. Gähler, 2-metrische Räume und Ihre Topologische Struktur, Math. Nachr. 26 (1963) 115–148.
[20] S. Gähler, Lineare 2-normietre Räume, Math. Nachr. 28 (1964) 1–43.
[21] S. Gähler, Uber 2-Banach räume, Math. Nachr. 42 (1969) 335–347.
[22] White A. George Jr., 2-Banach spaces, Math. Nachr. 42 (1969) 43–60.
[23] S. Gähler, A.H. Siddiqi, S.C. Gupta, Contributions to non-archimedean functional analysis, Math. Nachr. 69 (1975) 162–171.
[24] A.H. Siddiqi, 2-normed spaces, Aligarh Bull. Math. (1980) 53–70.
[25] H. Gunawan, Mashadi, On finite dimensional 2-normed spaces, Soochow J. Math. 27 (3) (2001) 321–329.
[26] H. Gunawan, On n-inner products, n-norms, and the Cauchy–Schwarz inequality, Sci. Math. Jpn. 5 (2001) 47–54.
[27] H. Gunawan, Mashadi, On n-normed spaces, Int. J. Math. Sci. 27 (10) (2001) 631–639.
[28] H. Gunawan, The space of p-summable sequences and its natural n-norm, Bull. Aust. Math. Soc. 64 (1) (2001) 137–147.
[29] M. Gürdal, S. Pehlivan, The statistical convergence in 2-Banach spaces, Thai. J. Math. 2 (1) (2004) 107–113.
[30] M. Gürdal, μ-statistical characterization of completion of normed and inner product spaces, Int. J. Pure Appl. Math. 22 (4) (2005) 439–449.
[31] M. Gürdal, I. Açık, On $\mathcal{I}$-Cauchy sequences in 2-normed spaces, Math. Inequal. Appl. 11 (2) (2008) 349–354.
[32] M. Gürdal, A. Sahiner, I. Açık, Approximation theory in 2-Banach spaces, Nonlinear Anal. 71 (2009) 1654–1661.
[33] A. Şahiner, M. Gürdal, S. Saltan, H. Gunawan, Ideal convergence in 2-normed spaces, Taiwanese J. Math. 11 (4) (2007) 1477–1484.
[34] J.L. Kelley, General Topology, Springer-Verlag, New York, 1955.
[35] C. Kuratowski, Topology I, PWN, Warszawa, 1958.
[36] P. Kostyrko, M. Macaj, T. Salat, M. Sleziak, $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points, Math. Slovaca 55 (2005) 443–464.
[37] K. Dems, On $\mathcal{I}$-Cauchy sequences, Real Anal. Exchange 3 (2004–2005) 123–128.
[38] A. Nabiev, S. Pehlivan, M. Gürdal, On $\mathcal{I}$-Cauchy sequences, Taiwanese J. Math. 11 (2) (2007) 569–576.
[39] W.R. Freese, Y. Cho, Geometry of Linear 2-normed Spaces, Nova Science Publishers, Huntington, NY, 2001.
[40] S. Gähler, Untersuchungen uber verallgemeinerte m-metrische Räume. II, Math. Nachr. 40 (1969) 229–264.